

Control of Multiple-Input, Multiple-Output (MIMO) Processes

- 18.1 Process Interactions and Control Loop Interactions
- 18.2 Pairing of Controlled and Manipulated Variables
- 18.3 Singular Value Analysis
- 18.4 Tuning of Multiloop PID Control Systems
- 18.5 Decoupling and Multivariable Control Strategies
- 18.6 Strategies for Reducing Control Loop Interactions

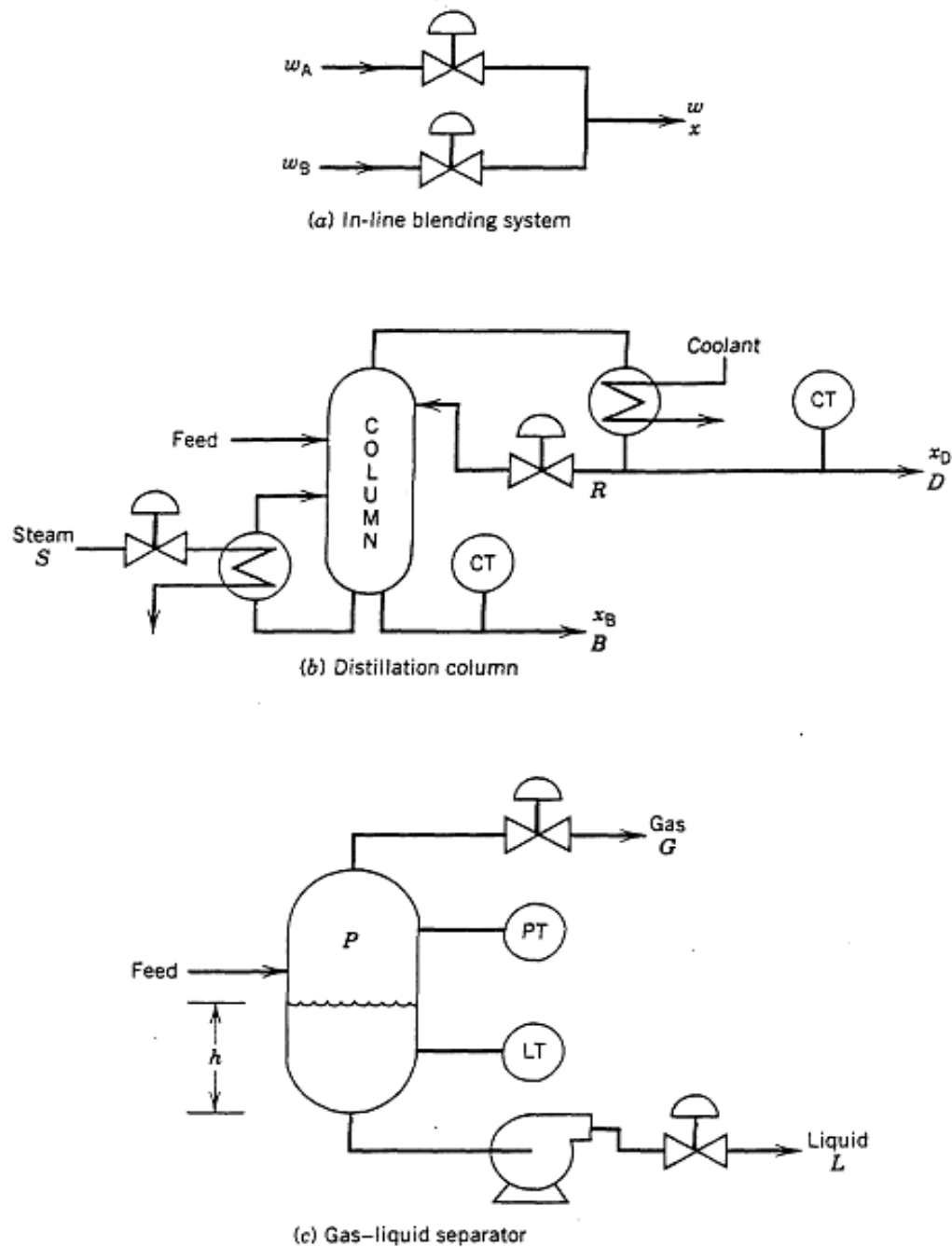
Control of Multivariable Processes

- In practical control problems there typically are a number of process variables which must be controlled and a number which can be manipulated.

Example: product quality and throughput must usually be controlled.

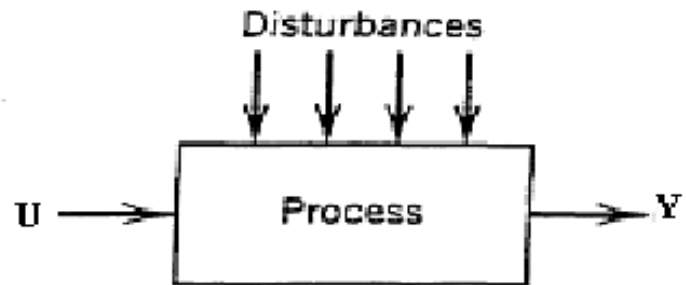
- Several simple physical examples are shown in Fig. 18.1.

Note the "process interactions" between controlled and manipulated variables.

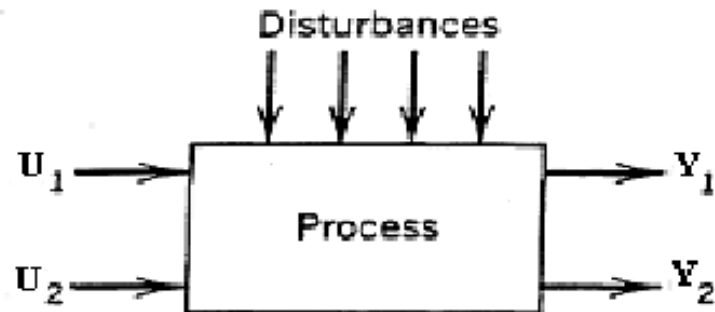


**SEE FIGURE 18.1
in text.**

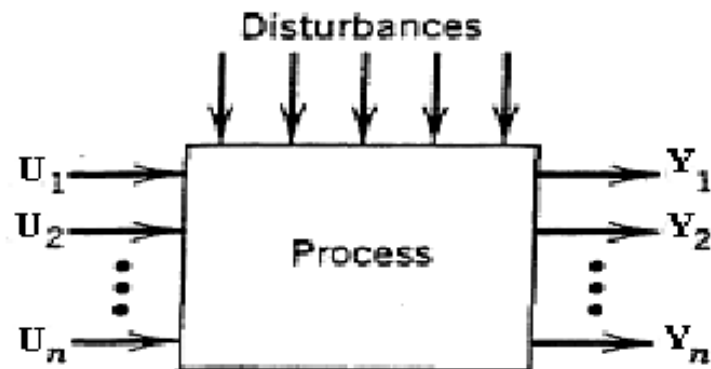
Figure 18.1. Physical examples of multivariable control problems.



(a) Single-input, single-output process with multiple disturbances



(b) Multiple-input, multiple-output process (2×2)



(c) Multiple-input, multiple-output process ($n \times n$)

Figure 18.2. SISO and MIMO control problems.

- **Controlled Variables:** $x_D, x_B, P, h_D,$ and h_B
- **Manipulated Variables:** $D, B, R, Q_D,$ and Q_B

Note: Possible multiloop control strategies = $5! = 120$

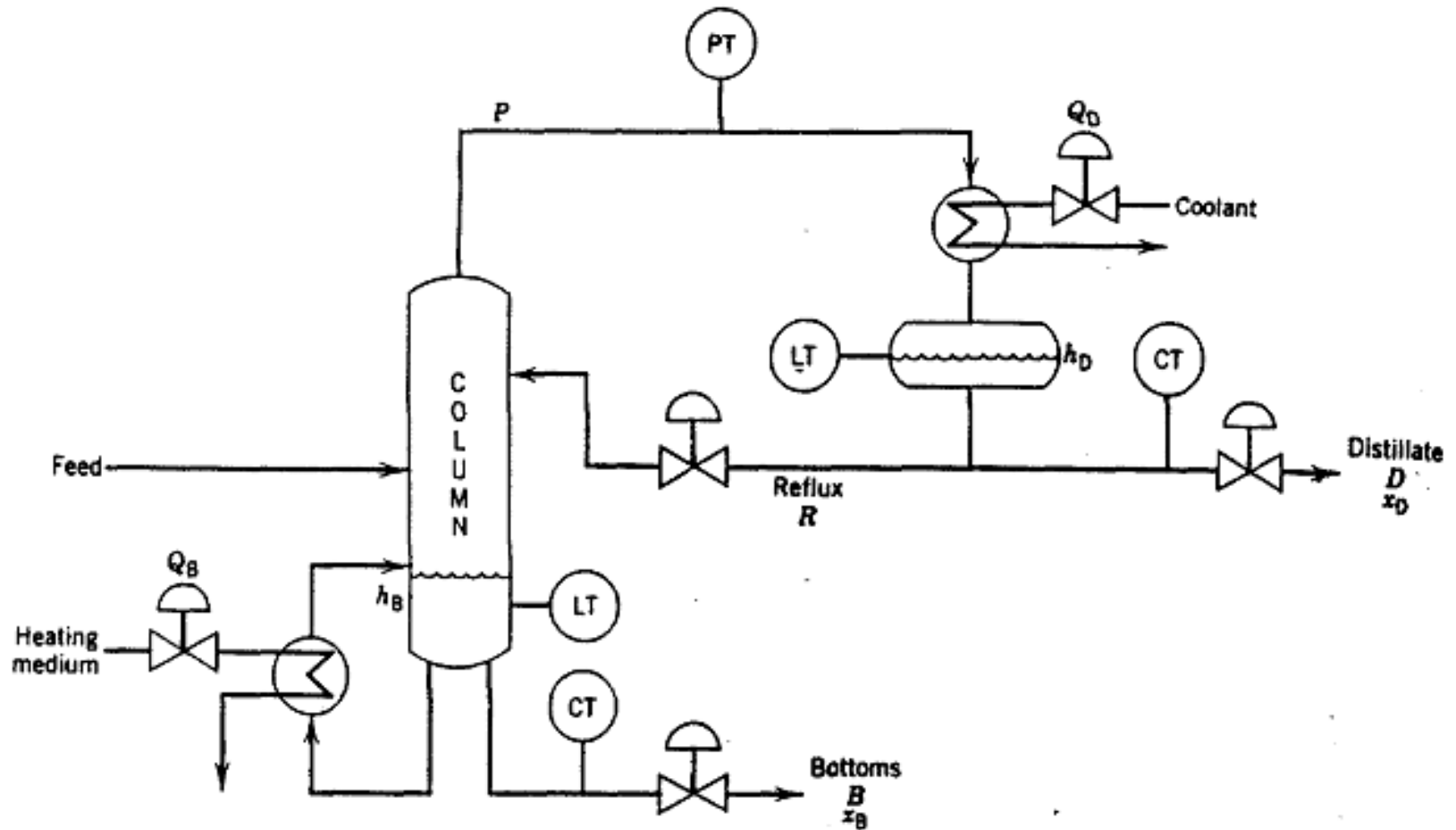


Figure 18.8. Controlled and manipulated variables for a typical distillation column.

- In this chapter we will be concerned with characterizing process interactions and selecting an appropriate multiloop control configuration.
- If process interactions are significant, even the best multiloop control system may not provide satisfactory control.
- In these situations there are incentives for considering multivariable control strategies.

Definitions:

- **Multiloop control:** Each manipulated variable depends on only a single controlled variable, i.e., a set of conventional feedback controllers.
- **Multivariable Control:** Each manipulated variable can depend on two or more of the controlled variables.

Examples: decoupling control, model predictive control

Multiloop Control Strategy

- Typical industrial approach
- Consists of using n standard FB controllers (e.g., PID), one for each controlled variable.
- **Control system design**
 1. Select controlled and manipulated variables.
 2. Select pairing of controlled and manipulated variables.
 3. Specify types of FB controllers.

Example: 2 x 2 system



Two possible controller pairings:

U_1 with Y_1 , U_2 with Y_2 (1-1/2-2 pairing)

or

U_1 with Y_2 , U_2 with Y_1 (1-2/2-1 pairing)

Note: For $n \times n$ system, $n!$ possible pairing configurations.

Transfer Function Model (2 x 2 system)

Two controlled variables and two manipulated variables
(4 transfer functions required)

$$\begin{aligned} \frac{Y_1(s)}{U_1(s)} = G_{P11}(s), & \quad \frac{Y_1(s)}{U_2(s)} = G_{P12}(s) \\ \frac{Y_2(s)}{U_1(s)} = G_{P21}(s), & \quad \frac{Y_2(s)}{U_2(s)} = G_{P22}(s) \end{aligned} \quad (18-1)$$

Thus, the input-output relations for the process can be written as:

$$Y_1(s) = G_{P11}(s)U_1(s) + G_{P12}(s)U_2(s) \quad (18-2)$$

$$Y_2(s) = G_{P21}(s)U_1(s) + G_{P22}(s)U_2(s) \quad (18-3)$$

Or in vector-matrix notation as,

$$\mathbf{Y}(s) = \mathbf{G}_p(s) \mathbf{U}(s) \quad (18-4)$$

where $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ are vectors,

$$\mathbf{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} \quad \mathbf{U}(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \quad (18-5)$$

And $\mathbf{G}_p(s)$ is the transfer function matrix for the process

$$\mathbf{G}_p(s) = \begin{bmatrix} G_{P11}(s) & G_{P12}(s) \\ G_{P21}(s) & G_{P22}(s) \end{bmatrix} \quad (18-6)$$

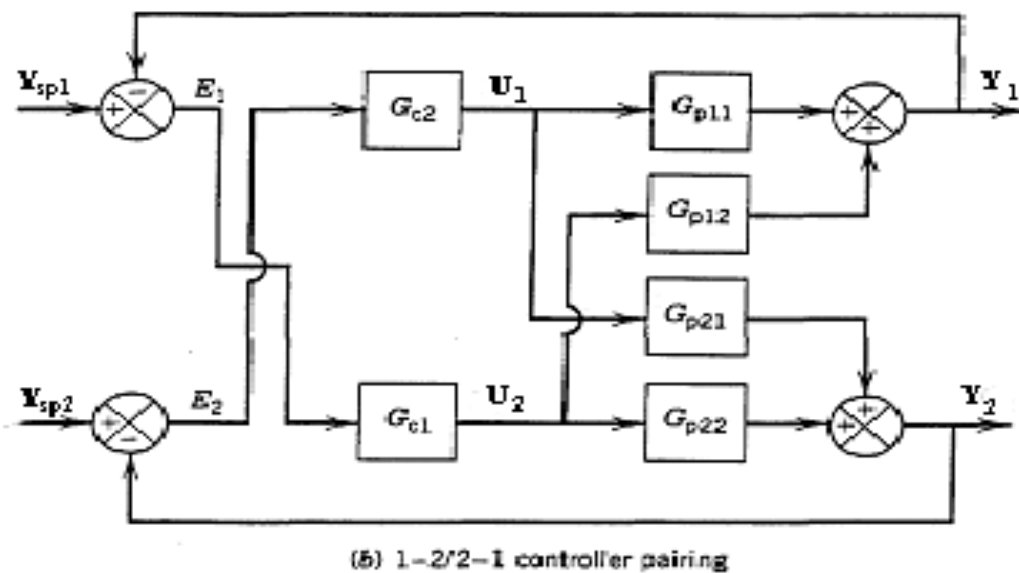
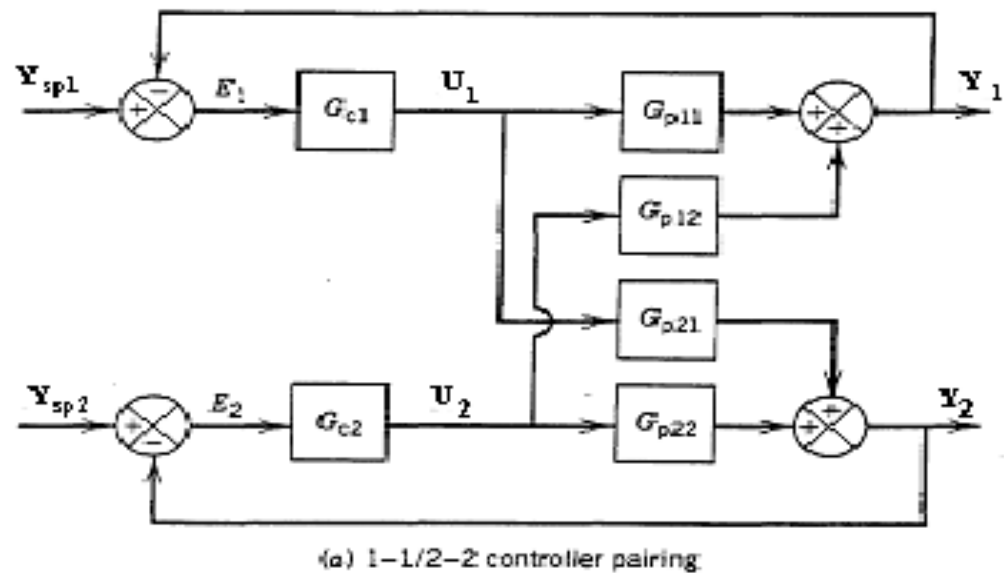


Figure 18.3. Block diagrams for 2 X 2 multiloop control schemes

Control-loop Interactions

- Process interactions may induce undesirable interactions between two or more control loops.

Example: 2 x 2 system

Control loop interactions are due to the presence of a third feedback loop.

- Problems arising from control loop interactions
 - i. Closed-loop system may become destabilized.
 - ii. Controller tuning becomes more difficult.

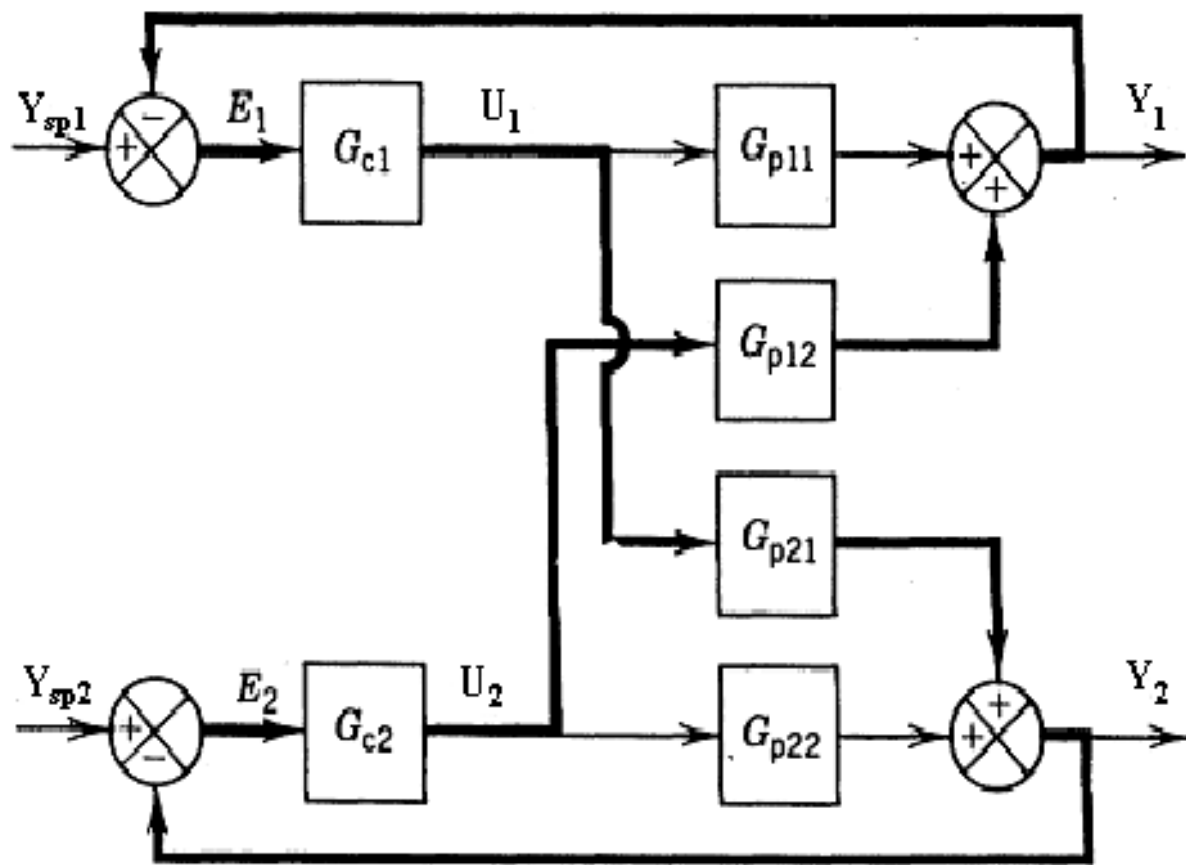


Figure 18.4. The hidden feedback control loop (in dark lines) for a 1-1/2-2 controller pairing.

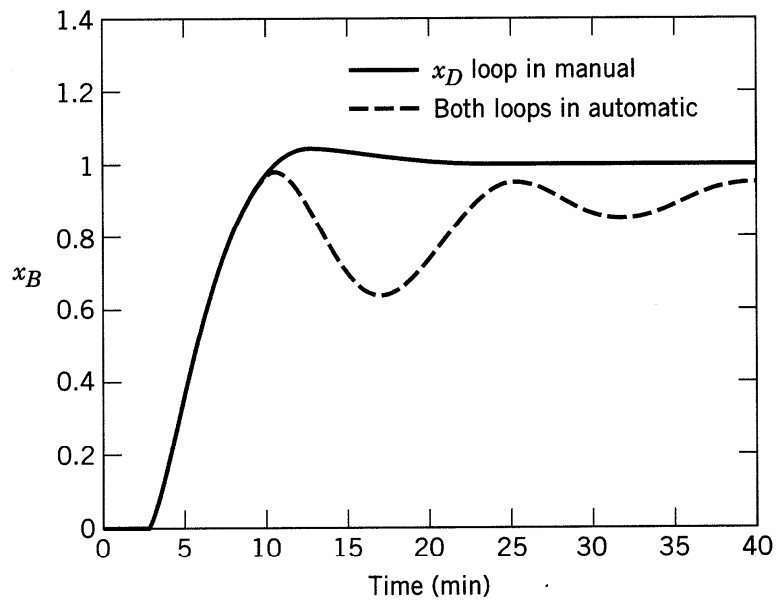
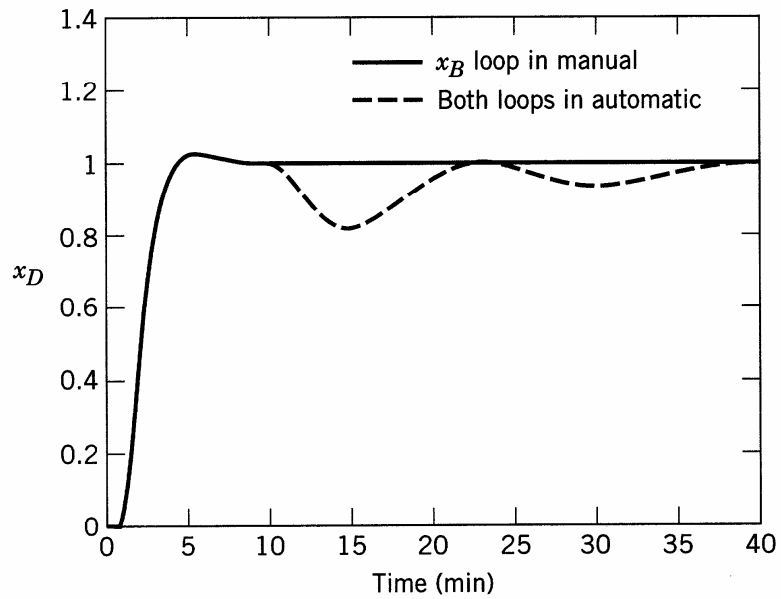


Figure 18.5 Set-point responses for Example 18.1 using ITAE tuning.

- $Y_1 = G_{p_{11}}U_1 + G_{p_{12}}U_2$
- $Y_2 = G_{p_{21}}U_1 + G_{p_{22}}U_2$
- If loop 2 is open, $Y_1 = G_{p_{11}}U_1$

Block Diagram Analysis

For the multiloop control configuration, the transfer function between a controlled and a manipulated variable depends on whether the other feedback control loops are open or closed.

Example: 2 x 2 system, 1-1/2 -2 pairing

From block diagram algebra we can show

$$\frac{Y_1(s)}{U_1(s)} = G_{P11}(s), \quad (\text{second loop open}) \quad (18-7)$$

$$\frac{Y_1(s)}{U_1(s)} = G_{P11} - \frac{G_{P12}G_{P21}G_{C2}}{1 + G_{C2}G_{P22}} \quad (\text{second loop closed}) \quad (18-11)$$

Note that the last expression contains G_{C2} .

18.1.2 Closed-Loop Stability

To evaluate the effects of control loop interactions further, again consider the block diagram for the 1-1/2-2 control scheme in Fig. 18.3a. Using block diagram algebra (see Chapter 11), we can derive the following expressions relating controlled variables and set points:

$$Y_1 = \Gamma_{11}Y_{sp1} + \Gamma_{12}Y_{sp2} \quad (18-13)$$

$$Y_2 = \Gamma_{21}Y_{sp1} + \Gamma_{22}Y_{sp2} \quad (18-14)$$

where the closed-loop transfer functions are

$$\Gamma_{11} = \frac{G_{c1}G_{p11} + G_{c1}G_{c2}(G_{p11}G_{p22} - G_{p12}G_{p21})}{\Delta(s)} \quad (18-15)$$

$$\Gamma_{12} = \frac{G_{c2}G_{p12}}{\Delta(s)} \quad (18-16)$$

$$\Gamma_{21} = \frac{G_{c1}G_{p21}}{\Delta(s)} \quad (18-17)$$

$$\Gamma_{22} = \frac{G_{c2}G_{p22} + G_{c1}G_{c2}(G_{p11}G_{p22} - G_{p12}G_{p21})}{\Delta(s)} \quad (18-18)$$

and $\Delta(s)$ is defined as

$$\Delta(s) = (1 + G_{c1}G_{p11})(1 + G_{c2}G_{p22}) - G_{c1}G_{c2}G_{p12}G_{p21} \quad (18-19)$$

Two important conclusions can be drawn from these closed-loop relations. First, a set-point change in one loop causes both controlled variables to change because Γ_{12} and Γ_{21} are not zero, in general. The second conclusion concerns the stability of the closed-loop system. Because each of the four closed-loop transfer functions in Eqs. 18-15 to 18-18 has the same denominator, the characteristic equation is $\Delta(s) = 0$, or

$$(1 + G_{c1}G_{p11})(1 + G_{c2}G_{p22}) - G_{c1}G_{c2}G_{p12}G_{p21} = 0 \quad (18-20)$$

Thus, the stability of the closed-loop system depends on both controllers, G_{c1} and G_{c2} , and all four process transfer functions. An analogous characteristic equation can be derived for the 1-2/2-1 control scheme in Fig. 18.3b.

For the special case where either $G_{p12} = 0$ or $G_{p21} = 0$, the characteristic equation in Eq. 18-20 reduces to

$$(1 + G_{c1}G_{p11})(1 + G_{c2}G_{p22}) = 0 \quad (18-21)$$

For this situation, the stability of the overall system merely depends on the stability of the two individual feedback control loops and their characteristic equations.

$$1 + G_{c1}G_{p11} = 0 \quad \text{and} \quad 1 + G_{c2}G_{p22} = 0 \quad (18-22)$$

Relative Gain Array

- **Provides two types of useful information:**
 1. Measure of process interactions
 2. Recommendation about best pairing of controlled and manipulated variables.
- **Requires knowledge of steady-state gains but not process dynamics.**

Example of RGA Analysis: 2 x 2 system

- Steady-state process model,

$$y_1 = K_{11}u_1 + K_{12}u_2$$

$$y_2 = K_{21}u_1 + K_{22}u_2$$

The RGA, Λ , is defined as:

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

where the relative gain, λ_{ij} , relates the i^{th} controlled variable and the j^{th} manipulated variable

$$\lambda_{ij} = \frac{\left(\partial y_i / \partial u_j\right)_u}{\left(\partial y_i / \partial u_j\right)_y} = \frac{\text{open-loop gain}}{\text{closed-loop gain}} \quad (18-24)$$

Scaling Properties:

- i. λ_{ij} is dimensionless
- ii. $\sum_i \lambda_{ij} = \sum_j \lambda_{ij} = 1.0$

For a 2 x 2 system,

$$\lambda_{11} = \frac{1}{1 - \frac{K_{12}K_{21}}{K_{11}K_{22}}}, \quad \lambda_{12} = 1 - \lambda_{11} = \lambda_{21} \quad (18-34)$$

Recommended Controller Pairing

It corresponds to the λ_{ij} which have the largest positive values that are closest to one.

In general:

1. Pairings which correspond to negative pairings should not be selected.
2. Otherwise, choose the pairing which has λ_{ij} closest to one.

Examples:

Process Gain
Matrix, \mathbf{K} :

Relative Gain
Array, $\mathbf{\Lambda}$:

$$\begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & K_{12} \\ K_{21} & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For 2 x 2 systems:

$$y_1 = K_{11}u_1 + K_{12}u_2 \quad \lambda_{11} = \frac{1}{1 - \frac{K_{12}K_{21}}{K_{11}K_{22}}}, \quad \lambda_{12} = 1 - \lambda_{11} = \lambda_{21}$$

$$y_2 = K_{21}u_1 + K_{22}u_2$$

Example 1:

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} 2 & 1.5 \\ 1.5 & 2 \end{bmatrix}$$

∴

$$\therefore \mathbf{A} = \begin{bmatrix} 2.29 & -1.29 \\ -1.29 & 2.29 \end{bmatrix}$$

Recommended pairing is Y_1 and U_1 , Y_2 and U_2 .

Example 2:

$$\mathbf{K} = \begin{bmatrix} -2 & 1.5 \\ 1.5 & 2 \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 0.64 & 0.36 \\ 0.36 & 0.64 \end{bmatrix}$$

∴

Recommended pairing is Y_1 with U_1 and Y_2 with U_2 .

RGA for Higher-Order Systems

For and $n \times n$ system,

$$\mathbf{A} = \begin{matrix} & \begin{matrix} u_1 & u_2 & \cdots & u_n \end{matrix} \\ \begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{matrix} & \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} \end{matrix} \quad (18-25)$$

Each λ_{ij} can be calculated from the relation,

$$\lambda_{ij} = K_{ij} H_{ij} \quad (18-37)$$

where K_{ij} is the (i,j) -element of the steady-state gain \mathbf{K} matrix,

$$\mathbf{y} = \mathbf{K}\mathbf{u}$$

H_{ij} is the (i,j) -element of the $\mathbf{H} = (\mathbf{K}^{-1})^T$

Note :

$$\mathbf{A} \neq \mathbf{KH}$$

Example: Hydrocracker

The RGA for a hydrocracker has been reported as,

$$\mathbf{A} = \begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \left[\begin{array}{cccc} 0.931 & 0.150 & 0.080 & -0.164 \\ -0.011 & -0.429 & 0.286 & 1.154 \\ -0.135 & 3.314 & -0.270 & -1.910 \\ 0.215 & -2.030 & 0.900 & 1.919 \end{array} \right] \end{array}$$

Recommended controller pairing?

Singular Value Analysis

- Any real $m \times n$ matrix can be factored as,

$$\mathbf{K} = \mathbf{W} \mathbf{\Sigma} \mathbf{V}^T$$

- Matrix $\mathbf{\Sigma}$ is a diagonal matrix of singular values:

$$\mathbf{\Sigma} = \text{diag} (\sigma_1, \sigma_2, \dots, \sigma_r)$$

- The singular values are the positive square roots of the eigenvalues of $\mathbf{K}^T \mathbf{K}$ ($r =$ the rank of $\mathbf{K}^T \mathbf{K}$).
 - The columns of matrices \mathbf{W} and \mathbf{V} are *orthonormal*. Thus,
- $$\mathbf{W}\mathbf{W}^T = \mathbf{I} \quad \text{and} \quad \mathbf{V}\mathbf{V}^T = \mathbf{I}$$
- Can calculate $\mathbf{\Sigma}$, \mathbf{W} , and \mathbf{V} using MATLAB command, *svd*.
 - *Condition number (CN)* is defined to be the ratio of the largest to the smallest singular value,

$$CN = \frac{\sigma_1}{\sigma_r}$$

- A large value of *CN* indicates that \mathbf{K} is ill-conditioned.

Condition Number

- CN is a measure of sensitivity of the matrix properties to changes in individual elements.
- Consider the RGA for a 2x2 process,

$$\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix} \Rightarrow \mathbf{A} = \mathbf{I}$$

- If K_{12} changes from 0 to 0.1, then \mathbf{K} becomes a singular matrix, which corresponds to a process that is difficult to control.
- RGA and SVA used together can indicate whether a process is easy (or difficult) to control.

$$\Sigma(\mathbf{K}) = \begin{bmatrix} 10.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad \text{CN} = 101$$

- \mathbf{K} is poorly conditioned when CN is a large number (e.g., > 10). Thus small changes in the model for this process can make it very difficult to control.

Selection of Inputs and Outputs

- Arrange the singular values in order of largest to smallest and look for any $\sigma_i/\sigma_{i-1} > 10$; then one or more inputs (or outputs) can be deleted.
- Delete one row and one column of \mathbf{K} at a time and evaluate the properties of the reduced gain matrix.
- **Example:**

$$\mathbf{K} = \begin{bmatrix} 0.48 & 0.90 & -0.006 \\ 0.52 & 0.95 & 0.008 \\ 0.90 & -0.95 & 0.020 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.5714 & 0.3766 & 0.7292 \\ 0.6035 & 0.4093 & -0.6843 \\ -0.5561 & 0.8311 & 0.0066 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1.618 & 0 & 0 \\ 0 & 1.143 & 0 \\ 0 & 0 & 0.0097 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.0541 & 0.9984 & 0.0151 \\ 0.9985 & -0.0540 & -0.0068 \\ -0.0060 & 0.0154 & -0.9999 \end{bmatrix}$$

•

$$\text{CN} = 166.5 (\sigma_1/\sigma_3)$$

The RGA is:

$$A = \begin{bmatrix} -2.4376 & 3.0241 & 0.4135 \\ 1.2211 & -0.7617 & 0.5407 \\ 2.2165 & -1.2623 & 0.0458 \end{bmatrix}$$

Preliminary pairing: y_1-u_2 , y_2-u_3 , y_3-u_1 .

CN suggests only two output variables can be controlled. Eliminate one input and one output ($3 \times 3 \rightarrow 2 \times 2$).

Table 18.3 CN and λ for Different 2×2 Pairings, Example 18.7

Pairing Number	Controlled Variables	Manipulated Variables	CN	λ
1	y_1, y_2	u_1, u_2	184	39.0
2	y_1, y_2	u_1, u_3	72.0	0.552
3	y_1, y_2	u_2, u_3	133	0.558
4	y_1, y_3	u_2, u_1	1.51	0.640
5	y_1, y_3	u_1, u_3	69.4	0.640
6	y_1, y_3	u_2, u_3	139	1.463
7	y_2, y_3	u_2, u_1	1.45	0.634
8	y_2, y_3	u_1, u_3	338	3.25
9	y_2, y_3	u_2, u_3	67.9	0.714

Alternative Strategies for Dealing with Undesirable Control Loop Interactions

1. "Detune" one or more FB controllers.
2. Select different manipulated or controlled variables.
e.g., nonlinear functions of original variables
3. Use a decoupling control scheme.
4. Use some other type of multivariable control scheme.

Decoupling Control Systems

- **Basic Idea:** Use additional controllers to compensate for process interactions and thus reduce control loop interactions
- Ideally, decoupling control allows setpoint changes to affect only the desired controlled variables.
- Typically, decoupling controllers are designed using a simple process model (e.g., a steady-state model or transfer function model)

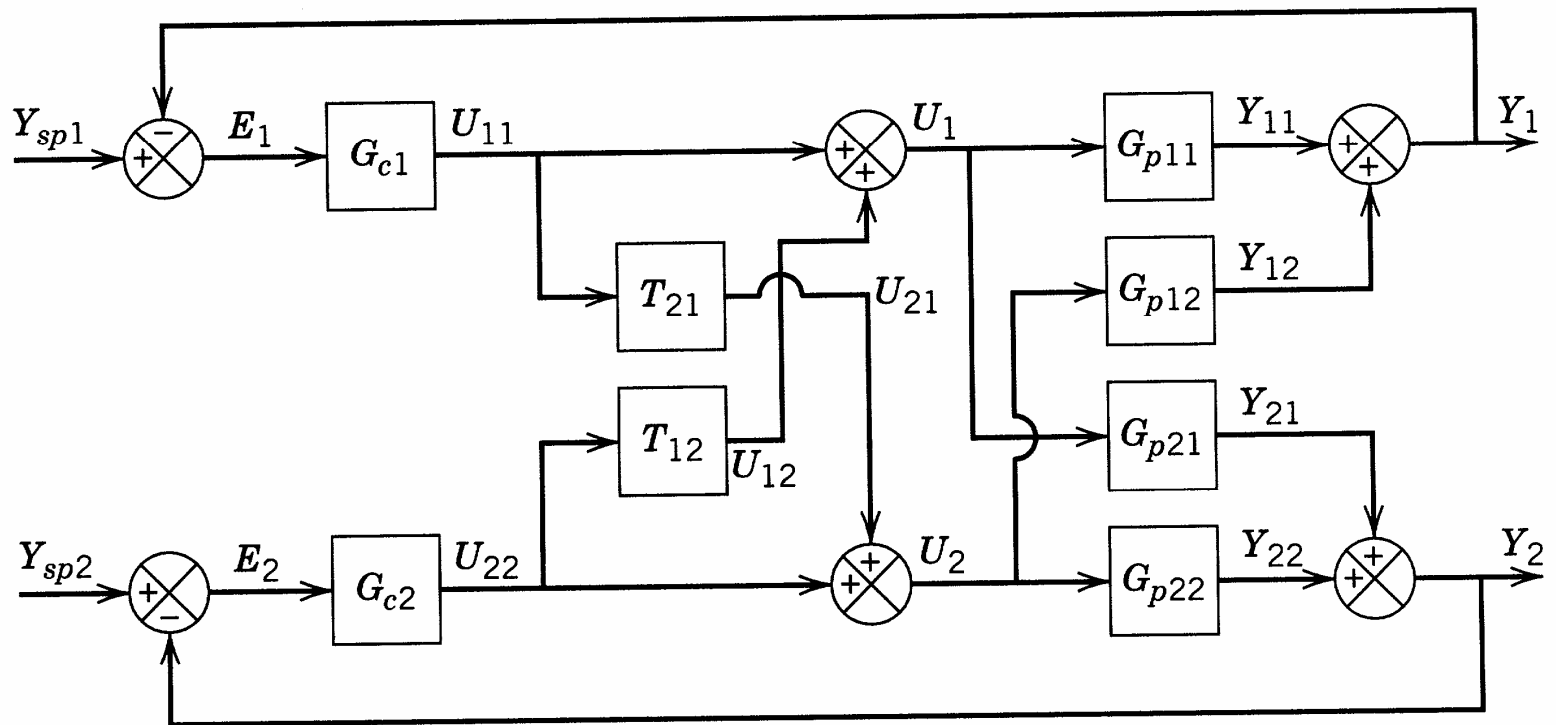


Figure 18.9 A decoupling control system.

Decoupler Design Equations

We want cross-controller, T_{12} , to cancel the effect of U_2 on Y_1 . Thus, we would like,

$$T_{12}G_{P11}U_{22} + G_{P12}U_{22} = 0 \quad (18-79)$$

Because $U_{22} \neq 0$ in general, then

$$T_{12} = -\frac{G_{P12}}{G_{P11}} \quad (18-80)$$

Similarly, we want T_{21} to cancel the effect of U_1 on Y_2 . Thus, we require that,

$$T_{21}G_{P22}U_{11} + G_{P21}U_{11} = 0 \quad (18-76)$$

$$\therefore T_{21} = -\frac{G_{P21}}{G_{P22}} \quad (18-78)$$

Compare with the design equations for feedforward control based on block diagram analysis

Variations on a Theme

1. *Partial Decoupling:*

Use only one “cross-controller.”

2. *Static Decoupling:*

Design to eliminate SS interactions

Ideal decouplers are merely gains:

$$T_{12} = -\frac{K_{P12}}{K_{P11}} \quad (18-85)$$

$$T_{21} = -\frac{K_{P21}}{K_{P22}} \quad (18-86)$$

3. *Nonlinear Decoupling*

Appropriate for nonlinear processes.